

Old and New Facets of y -Scaling: the Universal Features of Nuclear Structure Functions and Nucleon Momentum Distributions

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Some systematic general features of y -scaling structure functions, which are essentially independent of detailed dynamics, are pointed out. Their physical interpretation in terms of general characteristics, such as a mean field description and nucleon-nucleon correlations, is given and their relationship to the momentum distributions illustrated. A new scaling variable is proposed which allows a direct expression of the scaling function in term of an integral over the momentum distributions thereby avoiding explicit consideration of binding corrections.

Inclusive quasi-elastic electron scattering is a powerful method for measuring the momentum distribution of nucleons inside a nucleus. This is most succinctly made manifest by expressing the data in terms of the scaling variable y which, over a large kinematic range, can be identified as the longitudinal momentum of the struck nucleon, k_{\parallel} [1]. At sufficiently large momentum transfers, \mathbf{q} , the structure function, $W(\nu, q^2)$, which represents the deviation of the cross-section from scattering from free nucleons, scales to a function of the single variable y according to $qW(\nu, q^2) \approx f(y)$ where ν is the electron energy loss and $q \equiv |\mathbf{q}|$. Thus, in the scaling limit, qW approaches a function that effectively traces out the longitudinal momentum distribution of the nucleons:

$$f(y) = \int n(k_{\parallel}, \mathbf{k}_{\perp}) d^2 \mathbf{k}_{\perp} = 2\pi \int_{|y|}^{\infty} n(k) k dk \quad (1)$$

Here, $n(k)$ is the conventional nucleon momentum distribution function normalized such that

$$\int d^3 k n(k) = \int_{-\infty}^{\infty} dy f(y) = 1 \quad (2)$$

Knowledge of $f(y)$ can therefore be used to obtain $n(k)$ by inverting Eq. (1):

$$n(k) = -\frac{1}{2\pi y} \frac{df(y)}{dy} \quad |y| = k \quad (3)$$

More generally, $qW(\nu, q^2)$ is related to the spectral function, $P(k, E)$, which depends on the energy (E) as well as the momentum of the nucleons [2]: $qW(\nu, q^2) \approx F(y) = f(y) - B(y)$ where

$$B(y) = 2\pi \int_{E_{min}}^{\infty} dE \int_{|y|}^{k_{min}(y, E)} P_1(k, E) \quad (4)$$

P_1 is that part of $P(k, E)$ generated by ground state correlations; (thus, in a mean field description or, for the case of 2H , $P_1 = 0$) [3].

Over the past several years there have been vigorous efforts to explore y -scaling over a wide range of nuclei [4]. The purpose of this paper is to point out, and give some insight into, some general universal features of $f(y)$ that are essentially independent of the detailed dynamics and which seem to have been overlooked in past discussions. We shall show that the overall structure and systematics of the data can, to a large extent, be understood in terms of some rather general characteristics of nuclei. We provide a physical interpretation of these features and show their relationship to momentum distributions. In addition, we shall illustrate a new approach which allows a determination of $f(y)$ directly from experimental data, thereby avoiding use of the theoretical “binding correction”, $B(y)$. Up to now, longitudinal momentum distributions have been obtained by extracting $F(y)$ from data and then estimating $B(y)$ theoretically. Based on such a procedure a systematic analysis, to be presented elsewhere [5], exhibits the following general features of $f(y)$ for nuclei with $A < 56$:

- i) $f(0)$ decreases monotonically with A , from $\sim 10 MeV^{-1}$ when $A = 2$ to $\sim 3 MeV^{-1}$ for heavy nuclei; moreover, for $y \sim 0$, $f(y) \sim (\alpha^2 + y^2)^{-1}$, with α ranging from $\sim 45 MeV$ for $A = 2$, to $\sim 140 MeV$ for $A = 56$.
- ii) For $50 MeV \leq |y| \leq 200 MeV$, $F(y) \sim e^{-a^2 y^2}$ with a ranging from $\sim 50 MeV$ for $A = 2$, to $\sim 150 MeV$ for $A = 56$.
- iii) For $|y| \geq 400 MeV$, $f(y) \sim B e^{-b|y|}$, with B ranging from $2.5 \times 10^{-4} MeV^{-1}$ for $A = 2$, to $2 \times 10^{-4} MeV^{-1}$ for $A = 56$, and, most intriguingly, $b = 6 \times 10^{-3} MeV^{-1}$, independent of A .

The following simple form for $f(y)$ yields an excellent representation of these general features:

$$f(y) = \frac{Ae^{-a^2y^2}}{\alpha^2 + y^2} + Be^{-b|y|} \quad (5)$$

The first term dominates the small y -behavior whereas the second term dominates large y . Let us now discuss the motivation and interpretation of this equation. The systematics of the first term are determined by the small and intermediate momentum behaviour of the single particle wave function. For $|y| \leq \alpha$ this can be straightforwardly understood in terms of a zero range type of approximation and is, therefore, insensitive to the details of the microscopic dynamics, or of a specific model. The long distance behavior of a single particle wave function is controlled by its separation energy ($Q \equiv M + M_{A-1} - M_A$) and is given by $\frac{e^{-\alpha r}}{r}$ where $\alpha = (2\mu Q)^{\frac{1}{2}}$, μ being the reduced mass of the nucleon. In momentum space (see Eq. (6) below) this translates into $(k^2 + \alpha^2)^{-1}$ so the parameter α occurring in (5) is to be identified with $(2\mu Q)^{\frac{1}{2}}$. This agrees well with fits to the data summarized in (i) above.

Before discussing the intermediate range it is instructive to consider first the large y -behavior. Perhaps the most intriguing phenomenological characteristic of the data is that $f(y)$ falls off exponentially at large y with a similar slope parameter for all nuclei, including the deuteron. Unlike the behavior for small y there is, as far as we are aware, no simple argument to explain this remarkable fact. Since (i) b is almost the same for all nuclei including $A = 2$, i.e., $f(y)$, at large y , appears to be simply the rescaled longitudinal scaling function of the deuteron; and (ii) $b(\approx 1.18 \text{ fm}) \ll 1/\alpha_D(\approx 4.35 \text{ fm})$, we conclude that the term $e^{-b|y|}$ is related to the short range part of the deuteron wave function and reflects the universal nature of NN correlations in nuclei. In momentum space it can be related to the effective single-particle potential, $V(k)$, by

$$(k^2 + \alpha^2)\Psi(\mathbf{k}) = 2M \int \frac{d^3k'}{(2\pi)^3} V(\mathbf{k} - \mathbf{k}')\Psi(\mathbf{k}') \quad (6)$$

where $\Psi(\mathbf{k})$ is the single-particle wave function; ($n(k) = |\Psi(\mathbf{k})|^2$). The relationship of the exponential fall-off to $V(\mathbf{k})$ will be discussed in some detail in a later paper. Notice, incidentally, that once B and b are determined by fitting the large y data, the remaining two parameters in Eq. (5), A and a , can be fixed, respectively, from the value of $f(0)$ and the normalization condition, Eq. (2). It is important to stress that, once this is done, there are no adjustable parameters for different nuclei.

The intermediate range is clearly sensitive to the parameter a , the gaussian form being dictated by the usual harmonic oscillator potential used in the shell model. Notice, however, that the gaussian is modulated by the correct $|y| < \alpha$ behaviour, namely $(y^2 + \alpha^2)^{-1}$, thereby ensuring that the wave function has the correct asymptotics. As an example, Fig. 1 shows the experimental

scaling functions, $f(y)$, for 2H and 4He compared to Eq. (5). The fit is excellent; for 4He , the value of a is slightly smaller than the conventional one obtained from a pure harmonic oscillator potential since the rms radius receives an additional contribution from the term $(\alpha^2 + y^2)^{-1}$ [5]. A systematic analysis for a large body of nuclei exhibiting the same features as those shown in Fig. 1 will be presented elsewhere.

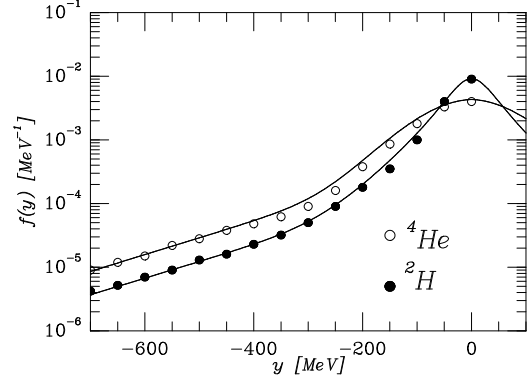


Fig. 1. The "experimental" longitudinal momentum distributions of 2H [2] and 4He [5] compared with Eq. (5) with $\alpha = 0.23 \text{ fm}^{-1}$ and $B = 0.048 \text{ fm}$ for 2H and $\alpha = 0.85 \text{ fm}^{-1}$ and $B = 0.12 \text{ fm}$ for 4He . For both nuclei $b = 1.2 \text{ fm}$. Note that the errors on $f(y)$ for 2H are of the same size as the data points; for 4He they are somewhat larger [5]

With these observations it is now possible to understand the normalization and evolution of $f(y)$ with A . First note that Eq. (1) implies

$$f(0) = \frac{1}{2} \int d^3k \frac{n(k)}{k} \quad (7)$$

In other words, $f(0)$ is a measure of $\langle 1/2k \rangle$ and so, as expected, is sensitive to the small momentum, or large distance, behavior of the wave function. Now, typical mean momenta vary from around 50 MeV for the deuteron up to almost 300 MeV for nuclear matter. We can, therefore, immediately see why $f(0)$ varies from around 10 for the deuteron down to around 2-3 for heavy nuclei [6]. Since $f(y)$ is constrained by a sum rule, Eq. (2), whose normalization is independent of the nucleus, a decrease in $f(0)$ as one changes the nucleus must be compensated for by a spreading of the curve for larger values of y . Thus, an understanding of $f(y)$ for small y coupled with an approximately universal fall-off for large y , together with the constraint of the sum rule, leads to an almost model-independent understanding of the gross features of the data for all nuclei.

To sum up, the "experimental" longitudinal momentum distribution can be thought of as the incoherent sum of a mean field contribution, ($f_0 = \frac{A}{\alpha^2 + y^2} e^{-a^2 y^2}$), with the correct model-independent small y -behaviour built in, and a "universal" correlation contribution ($f_1 =$

$Be^{-b|y|}$). Thus, the momentum distribution, $n(k)$, which is obtained from (3), is also a sum of two contributions: $n = n_0 + n_1$. This allows a comparison with results from many body calculations in which n_0 and n_1 have been separately calculated. Of particular relevance are not only the shapes of n_0 and n_1 , but also their normalizations, $S_{0(1)} \equiv \int n_{0(1)} d^3k = \int f_{0(1)} dy$, the so called occupation probabilities, which, theoretically, turn out to be, for ${}^4\text{He}$, $S_0 \sim 0.8$ and $S_1 \sim 0.2$ [7] whereas Eq. (5) yields $S_0 = 0.76$ and $S_1 = 0.24$ [8]. A comparison between the momentum distributions obtained from y -scaling and the theoretical ones is shown in Fig. 2. As can be seen the $n(k)$ compare very well with theoretical calculations [9]. Thus, the physical interpretation of our simple parametrization of $f(y)$ strongly suggests a two-component form consisting of mean field and correlation contributions. In addition, we have shown that the momentum distributions resulting from y -scaling agree with theoretical ones (in agreement with previous results from Ref. [2]).

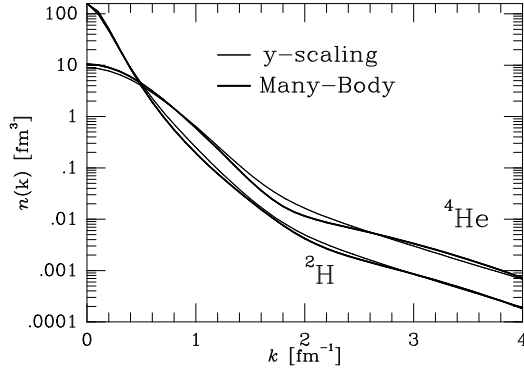


Fig. 2. The nucleon momentum distributions for ${}^2\text{H}$ and ${}^4\text{He}$ resulting from Eq. (5), compared with realistic momentum distributions (as parametrized in ref. [11](b))

In making comparisons with data and/or extracting parameters for fits which can be compared to theoretical models a problem arises that $f(y)$ is affected by large errors, particularly at large y . These are due to (a) experimental errors and lack of good inclusive data at large $Q^2 (\equiv \nu^2 - \mathbf{q}^2)$ and large $x (\equiv Q^2/2M\nu) \geq 1$ and, (b) more to the point here, the procedure used for applying the theoretical correction, $B(y)$. The first source of error could in principle be minimized by the new generation of inclusive data expected from CEBAF [10]. Here, we would like to tackle the second problem by presenting a new approach which would allow a determination of $f(y)$ essentially free from any theoretical correction contamination.

The binding correction arises from the fact that the scaling variable y is effectively obtained from energy conservation

$$\nu + M_A = [(M_{A-1} + E_{A-1}^*)^2 + \mathbf{k}^2]^{1/2} + [M^2 + (\mathbf{k} + \mathbf{q})^2]^{1/2} \quad (8)$$

by setting $k = y$, $\frac{\mathbf{k} \cdot \mathbf{q}}{kq} = \cos \alpha = 1$, and, most importantly, the excitation energy, $E_{A-1}^* = 0$; thus, y represents the nucleon longitudinal momentum corresponding to the minimum value of the removal energy ($E = E_{\min} + E_{A-1}^*$). The minimum value of the nucleon's momentum allowed by energy conservation in the limit $q \equiv |\mathbf{q}| \rightarrow \infty$, becomes $k_{\min}(y, E) = |y - (E - E_{\min})|$. It can be seen that only when $E = E_{\min}$ does $k_{\min}(y, E) = |y|$, in which case $B = 0$ and $F(y) = f(y)$. However, the final spectator ($A - 1$) system can be left in all possible excited states, including the continuum, so, in general, $E_{A-1}^* \neq 0$ and $E > E_{\min}$, so $B(y) \neq 0$, and $F(y) \neq f(y)$. Thus, it is the dependence of k_{\min} on E_{A-1}^* that gives rise to the binding correction.

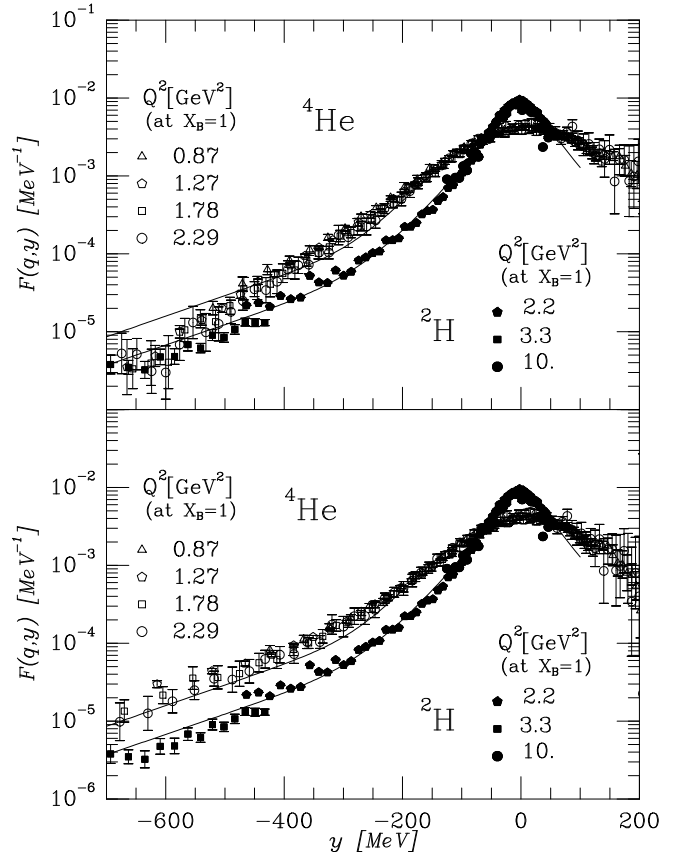


Fig. 3. The experimental scaling functions $F(y)$ for ${}^2\text{H}$ and ${}^4\text{He}$ compared with the longitudinal momentum distributions $f(y)$ given by Eq. (5). In the upper part of the figure data are plotted vs. y_1 whereas, in the lower part, vs. the new scaling variable, y_2 (see Eq. (9); for ${}^2\text{H}$, $y_1 = y_2$). Only the data at large q [13] were considered. As discussed in the text, scaling is greatly improved when the data are plotted vs. y_2 .

We propose to take account of this in the following way. We adhere to the widespread consensus [11] that the large k and E behaviours of the nuclear wave function are governed by configurations in which the high momentum of a correlated nucleon (1, say) is almost entirely balanced by another correlated nucleon (2, say), with the spectator ($A - 2$) system taking only a small fraction of k . Within such a picture, it can be shown [11] that $E_{A-1}^* \approx \frac{A-2}{A-1} \frac{k^2}{2M}$, which is nothing but the energy associated with the relative motion of nucleon 2 and system ($A - 2$). Such a relation represents the very physical phenomenon that in the continuum part of the spectral function, E is a function of k . When this is used in Eq. (8) a new scaling variable, y_2 , is obtained, which incorporates the excitation energy of the final ($A - 1$) system in the continuum [12]:

$$y_2 = \left| -\frac{q}{2} + \left[\frac{q^2}{4} - \frac{4\tilde{\nu}^2 M^2 - \tilde{W}^4}{\tilde{W}^2} \right]^{1/2} \right| \quad (9)$$

Here, $\tilde{\nu} = \nu + \tilde{M}$, $\tilde{M} = 2M - E_{th}^{(2)}$, $E_{th}^{(2)} = |E_A| - |E_{A-2}|$, and $\tilde{W}^2 = \tilde{M}^2 + 2\nu\tilde{M} - Q^2$. For the deuteron $E_{A-1}^* = 0$, so $y_2 \rightarrow y = | -q/2 + [q^2/4 - (4\tilde{\nu}^2 M^2 - \tilde{W}^4)/\tilde{W}^2]^{1/2} |$ with $\tilde{\nu} = \nu + M_d$ and $M = M_d$. Thus, in general, y_2 can be interpreted as the scaling variable pertaining to a “deuteron” with mass $\tilde{M} = 2M - E_{th}^{(2)}$. It is worth stressing that for small $y_2 (\ll (2MM_{A-1})^{1/2})$, $y_2 \approx y_1$, i.e. the usual variable is recovered. Thus y_2 is physically useful in both the correlation and the single particle regions. More importantly, since $k_{min}(q, \nu, E) \simeq |y_2|$, $B(y_2) \simeq 0$ so that $F(y_2) \simeq f(y_2)$. Thus, plotting data in terms of y_2 allows a *direct* determination of $f(y_2)$. If such a picture is correct, one would expect from our analysis above, the same behaviour of $f(y_2)$ at high values of y_2 for both the deuteron and complex nuclei which is, indeed, the case, as exhibited in Fig. 3. This is in contrast to what happens with $F(y)$.

We can summarise our conclusions as follows:

- i) The general universal features of the y -scaling function have been identified and interpreted in terms of three contributions: a model-independent zero-range contribution, a “universal” correlation contribution and a mean field contribution;
- ii) The shape and evolution of the curve have been understood both quantitatively and qualitatively on general grounds;
- iii) By defining a proper scaling variable which incorporates the excitation energy of the ($A - 1$) system generated by correlations, the longitudinal momentum distributions can be directly obtained from the experimental data, without introducing theoretical corrections.

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- [1] G. B. West, Phys. Rep. **18**, 263 (1975).
 - [2] C. Ciofi degli Atti, E. Pace and G. Salme, Phys. Rev. **C36**, 1208 (1987); *ibid* **C43**, 1155 (1991).
 - [3] Various corrections to this such as meson production, off-shell effects and so on, will not be considered in what follows since they do not play a central role in our discussion. These will be discussed in a later paper; some of these corrections, such as relativistic effects, can be reasonably well approximated by a suitable definition of y as in our discussion on binding corrections following Eq. (8) below.
 - [4] See, for example, D. B. Day *et al.*, Annu. Rev. Nucl. Part. Sci. **40**, 357 (1990).
 - [5] C. Ciofi degli Atti, D. Faralli and G. B. West, in preparation. This will also discuss errors associated with the parameters quoted here.
 - [6] This can be expressed analytically as follows: as already implied in the above discussion, since $B \ll A/\alpha^2$ and the second term falls off so rapidly with y , the normalization integral, Eq. (2), is dominated by the small y behaviour of $f(y)$, (i.e., by the first term in Eq. (5)). Thus, $f(0) \approx (\pi^{1/2}\alpha)^{-1} = (2\pi\mu Q)^{-1/2}$. Since the sum rule is more than 80% saturated by the first term this estimate should be good to 15 – 20% (see below). More importantly, however, it gives an excellent fit to the A -dependence of $f(0)$ and a simple explanation as to why $f(0) \approx 10$ for the deuteron (and not 1, say) whereas for iron it is ≈ 3 (and not 10).
 - [7] Y. Akaishi, Nucl. Phys. **A146**, 409 (1984); R. Schiavilla, V. R. Pandharipande and R. B. Wiringa, Nucl. Phys. **A449**, 219 (1986).
 - [8] It is worth emphasizing that our values for $S_{0(1)}$ are not adjustable parameters but result from a knowledge of $f(0)$ and the behaviour of $f(y)$ at large y coupled with the normalization condition, Eq. (2).
 - [9] Notice that Eq. (5) yields an $n(k)$ which has an unphysical singularity at $k = 0$. This is purely an artefact of our simple parametrization of $f(y)$ which can easily be removed by multiplying the second term in (5) by any function $g(y)$ that, for small y , $\approx (1 - by)$. A particularly convenient choice, which introduces no new parameters and has the added virtue that it introduces the correct $y \approx \alpha$ behaviour, is $(1 + by/(1 + y^2/\alpha^2))$. This has almost no effect on our fits. This, and other possibilities for removing the singularity, will be discussed in ref. [5]. Note that in Fig. 2 the singularity has been properly removed.
 - [10] CEBAF experiment 89 – 008, B. Filippone and D. Day, spokespersons.
 - [11] (a) L. L. Frankfurt and M. I. Strikman, Phys. Rep. **160**, 235, (1988); (b) C. Ciofi degli Atti and S. Simula, Phys. Rev. **C53**, 1686, (1996).
 - [12] From now on the usual scaling variable obtained from Eq. (8) will be denoted by y_1 .
 - [13] S. Rock *et al.*, Phys. Rev. Letts. **38**, 259, (1982) and references contained therein; D. Day *et al.*, Phys. Rev. Letts. **59**, 427 (1987).